

Black-Hole Thermodynamics and Renormalization

Dmitri V. Fursaev¹

*Laboratory of Theoretical Physics, Joint Institute for Nuclear Research,
Head Post Office, P.O.Box 79, Moscow, Russia*

and

*Dipartimento di Scienze Fisiche, Università di Napoli - Federico II -, and INFN
Sezione di Napoli, Mostra D'Oltremare Pad. 19, 80125, Napoli, Italy*

Abstract

Ultraviolet regime in quantum theory with horizons, contrary to ordinary theory, depends on the temperature of the system due to additional surface divergences in the effective action. We evaluate their general one-loop structure paying attention to effects of the curvature of the space-time near the horizon. In particular, apart from the area term, the entropy of a black hole is shown to acquire a topological correction in the form of the integral curvature of the horizon. To get the entropy, heat capacity and other thermodynamical quantities finite, such a kind of singularities should be removed by renormalization of a number of constants in a surface functional introduced in the effective action at arbitrary temperature. We also discuss a discrepancy in the different regularization techniques.

¹e-mail: fursaev@theor.jinrc.dubna.su

1 Introduction

Recently, starting from the t' Hooft's paper [1], much attention has been paid to one-loop computations of the entropy in quantum theory on the black-hole space-times (see [2]-[6] and references therein). Different methods employed on Rindler and Schwarzschild spaces have indicated a divergent surface term interpreted as quantum correction to the Bekenstein-Hawking entropy. Beside this, another term, not reducible to the horizon area, was pointed out in [6] for Schwarzschild hole. However, although a discussion of the particular cases is going on, the general geometrical form of the additional surface divergences, their renormalization and related physical aspects are not completely clear.

A formal reason for surface divergences on a black-hole space-time can be seen in the following way. The horizon surface is a set of fixed points with respect to the one-parameter isometry group associated with time translations. As a consequence, the corresponding finite-temperature quantum theory is formulated here in Euclidean space with conical defects near the horizon, which results to the surface divergences in the effective action similar to those in quantum theory with boundaries [7], [8].

An interesting feature of new divergences is in their dependence on temperature. This unusual effect is absent in ordinary finite-temperature theory where ultraviolet properties are determined by local geometry and are not sensible to choice of the quantum state [9]. The described features seem to have a universal character for the bifurcate Killing horizons, including the case of the cosmological ones [10]. A practical interest, however, is the Hawking temperature when the surface divergences of the effective action or free energy vanish identically. Remarkably, it does not exclude the divergent terms from their derivatives in temperature and in particular from the entropy.

There is no a unique point of view about the natural cut-off for such corrections. For instance, one opportunity is to take into account quantum fluctuations of the horizon [2], the other one is to use the superstring theory [4]. Thus, not excluding these possibilities, it seems to be useful to investigate how to remove the surface singularities in a way common for quantum field theory.

The aim of this letter is to represent a general structure of the surface terms for static

spaces with the horizons and to describe their renormalization. It will be done on the base of the heat kernel expansion near conical defects [11], [12]. This subject is considered in sections 2 and 3. Beside this, we evaluate a general form of the surface corrections to the black-hole entropy. Then we conclude with remarks on applicability of the different regularization schemes.

2 Divergences

We are interested in static space-times where thermal equilibrium can be well defined. In this case the free energy $F(\beta)$ and the one-loop effective action $W(\beta)$ in a scalar theory, including non-minimal coupling with the curvature, can be written as [9]

$$\beta F(\beta) = W(\beta) = \frac{1}{2} \log \det(-\square + \xi R + m^2) = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr} K_{M_\beta}(s) e^{-m^2 s} \quad (2.1)$$

where \square and $K_{M_\beta}(s)$ are the Laplace and the heat operators on a background manifold M_β being a Euclidean section of the corresponding static space. β is the inverse temperature of the system and is the period of the Euclidean time τ . With respect to the isometry group, associated with rotations in τ , the 2-dimensional surface Σ of the horizon is a set of the fixed points.

The Hawking temperature β_H^{-1} is determined by the surface gravity k as $\beta_H^{-1} = k/(2\pi)$. For the sake of simplicity we will assume that $\beta_H = 2\pi$ ($k = 1$), then its right value can be easily restored by changing β to $2\pi\beta/\beta_H$ in all the formulas. At the Hawking temperature M_{β_H} is a smooth manifold. However, if $\beta \neq \beta_H$, there is a conical singularity at Σ , although outside this surface, no matter how close, the geometry of M_β is the same as that of M_{β_H} . Hence, in comparing to M_{β_H} the scalar curvature acquires a delta-function contribution from Σ . Using for it the notation \bar{R} one can write

$$\int_{M_\beta} \bar{R} = 2(2\pi - \beta) \int_\Sigma + \int_{M_\beta} R \quad (2.2)$$

where R is the local curvature defined by the Riemann tensor in the smooth region and the additional surface term is provided by the conical singularity with deficit angle $2\pi - \beta$ [6],[13],[14]. Note, that for non-minimal coupling in (2.1) we imply the regular curvature R .

As was pointed out in [14], it is the integral (2.2) that should be used in the Euclidean gravitational action in quantum gravity. In particular the black hole is an extremum of (2.2) with subtracted boundary terms. Besides, the variations of the metric on Σ result to the condition $\beta = 2\pi$ corresponding to the smooth geometry.

The integral in the DeWitt-Schwinger representation (2.1) is known to diverge on the lower integration limit as $s \rightarrow 0$. The structure of this divergence can be immediately found from the asymptotic expansion of $Tr K_{M_\beta}(s)$, which reads [11],[12]

$$Tr K_{M_\beta}(s)|_{s \rightarrow 0} = \frac{1}{(4\pi s)^{d/2}} \sum_{n=0}^{\infty} (a_n + a_{\beta,n}) s^n \quad . \quad (2.3)$$

Here the standard heat coefficients a_n , $n \geq 1$, given by the integrals on the powers of the Riemann tensor and its derivatives,

$$a_1 = \left(\frac{1}{6} - \xi\right) \int_{M_\beta} R \quad , \quad (2.4)$$

$$a_2 = \int_{M_\beta} \left(\frac{1}{180} R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} - \frac{1}{6} \left(\frac{1}{5} - \xi\right) \square R + \frac{1}{2} \left(\frac{1}{6} - \xi\right)^2 R^2 \right) \quad , \dots, \quad (2.5)$$

get modified by the surface terms $a_{\beta,n}$ due to the conical singularities near Σ

$$a_{\beta,1} = \beta c_1(\beta) \int_{\Sigma} \quad , \quad (2.6)$$

$$a_{\beta,2} = \beta \int_{\Sigma} \left[c_1(\beta) \left(\left(\frac{1}{6} - \xi\right) R + \frac{1}{6} R_{\mu\nu} n_i^\mu n_i^\nu - \frac{1}{3} R_{\mu\nu\lambda\rho} n_i^\mu n_i^\lambda n_j^\nu n_j^\rho \right) + \right. \\ \left. c_2(\beta) \left(\frac{1}{2} R_{\mu\nu\lambda\rho} n_i^\mu n_i^\lambda n_j^\nu n_j^\rho - \frac{1}{4} R_{\mu\nu} n_i^\mu n_i^\nu \right) \right] \quad . \quad (2.7)$$

The components of the curvature tensor in (2.7) are taken on the surface Σ and n_i are two vectors orthogonal to Σ and normalized as $n_i^\mu n_{j\mu} = \delta_{ij}$. $c_{1,2}(\beta)$ in (2.6), (2.7) are defined as

$$c_1(\beta) = \frac{1}{6} \left(\left(\frac{2\pi}{\beta}\right)^2 - 1 \right) \quad , \quad c_2(\beta) = \frac{1}{15} c_1(\beta) \left(\left(\frac{2\pi}{\beta}\right)^2 + 11 \right) \quad . \quad (2.8)$$

All other surface coefficients $a_{\beta,n}$ have the structure similar to that of (2.6), (2.7).

The only coefficient to survive on the Rindler space is $a_{\beta,1}$ which is proportional to the area of the horizon. On the other hand, the next coefficient given by (2.7) depends on the curvature of M_β near Σ . In fact, $a_{\beta,2}$ is defined by three independent geometrical quantities: by orthogonal components of the Riemann and Ricci tensors, $R_{\mu\nu\lambda\rho} n_i^\mu n_i^\lambda n_j^\nu n_j^\rho$,

$R_{\mu\nu}n_i^\mu n_i^\nu$, and by the curvature R_Σ of the horizon surface. The latter enters in the scalar R in (2.7), which can be seen from the Gauss-Codacci equations [15]

$$R = R_\Sigma + 2R_{\mu\nu}n_i^\mu n_i^\nu - R_{\mu\nu\lambda\rho}n_i^\mu n_i^\lambda n_j^\nu n_j^\rho - (\chi_{i\mu}^\mu)^2 + \chi_i^{\mu\nu} \chi_{i\mu\nu} \quad (2.9)$$

where $\chi_i^{\mu\nu}$ are two second fundamental forms of Σ vanishing in the present case due to the isometry. Thus, for 4-dimensional space $a_{\beta,2}$ includes a pure topological term, the integral curvature of Σ being the Euler number. Moreover, as it follows from (2.9), for Schwarzschild black hole and, more generally, for Ricci flat 4-geometries, $R_{\mu\nu\lambda\rho}n_i^\mu n_i^\lambda n_j^\nu n_j^\rho = R_\Sigma$ and there will be only this topological term in $a_{\beta,2}$.

It is also interesting to find the expressions of these geometrical quantities for the Reissner-Nordstrom black hole with the mass M and charge $Q < M$

$$R_{\mu\nu}n_i^\mu n_i^\nu = -\frac{2r_-}{r_+^3} \quad , \quad R_{\mu\nu\lambda\rho}n_i^\mu n_i^\lambda n_j^\nu n_j^\rho = \frac{2r_+ - 4r_-}{r_+^3} \quad (2.10)$$

where, for the unit gravitational constant $G \equiv 1$,

$$r_\pm = M \pm (M^2 - Q^2)^{1/2} \quad (2.11)$$

are the radii of the outer and inner horizons and expressions (2.10) are evaluated at r_+ . So far as the horizon area is $4\pi r_+^2$ the surface coefficient $a_{\beta,2}$ includes the dimensionless terms proportional to the ratio r_-/r_+ . Note also, that the heat kernel expansion in the form (2.3) is applicable only when $Q < M$. However it fails for an extremal hole with $Q = M$ when singularity at Σ is not more conical. In this case the Hawking temperature β_H^{-1} is zero and polynomial coefficients $c_{1,2}(\beta)$, which really depend on dimensionless combinations $(\beta_H/\beta)^2$, turn out to be infinite at $\beta \neq \beta_H$.

The divergent part $W_{div}(\beta)$ of the effective action (2.1) can be easily obtained from (2.3). For example, in dimensional regularization [9] one can represent $W_{div}(\beta)$ as a sum of the volume, $W_{div, vol}$, and surface, $W_{div, surf}$, parts

$$W_{div}(\beta) = W_{div, vol}(\beta) + W_{div, surf}(\beta) \quad ,$$

$$W_{div, vol}(\beta) = -\frac{1}{32\pi^2\epsilon} \left[\frac{(m^2)^2}{2} a_0 - m^2 a_1 + a_2 \right] \quad , \quad (2.12)$$

$$W_{div, surf}(\beta) = -\frac{1}{32\pi^2\epsilon} \left[-m^2 a_{\beta,1} + a_{\beta,2} \right] \quad (2.13)$$

where ϵ is a regularization parameter interpreted as extra dimensions of the space. As it follows from (2.12), the volume integral $W_{div, vol}(\beta)$ represents standard ultraviolet divergences in quantum theory in curved space-time. Its dependence on temperature is trivial; β enters in $W_{div, vol}(\beta)$ as a multiplier. Hence, $W_{div, vol}(\beta)$ results to an infinite correction to the vacuum energy of the system, but does not contribute to its entropy. The properties of the functional $W_{div, surf}(\beta)$ are different. It is given on the horizon Σ and describes the surface divergences induced by conical singularities of M_β . Moreover, the equations show that in the free energy new divergences are the forth order polynomials in temperature and therefore they do contribute to the entropy. This interesting feature of quantum fields in presence of the horizons is absent in standard quantum theory.

3 Renormalization

In black-hole thermodynamics one is interested in the Hawking temperature ($\beta = 2\pi$) when M_β is a smooth manifold and all the surface terms vanish in the effective action. However, as follows from (2.6)-(2.8) and (2.13), the derivatives in β of $W_{surf, div}(\beta)$ are not equal to zero at $\beta = 2\pi$. Much attention has been paid to the fact that it results to the infinite corrections to the entropy of a black hole. For Rindler space-time these infinities can be removed from the entropy by renormalization of the gravitational constant [4]. For the Schwarzschild geometry the additional divergence due to the curvature can be eliminated by renormalization of the other coupling constant in the one-loop gravitational action [6].

These prescriptions do not seem to have a universal character so far as they only concern the entropy, but leave the divergences in the other physical quantities related with the higher derivatives in β of the effective action. For instance, the heat capacity depends on the second derivative and it can be shown to acquire the surface term at the Hawking temperature different from that of the entropy. It means that for complete solution of the renormalization problem one should renormalize the effective action at an *arbitrary* parameter β , which could provide then finite thermodynamical quantities.

It is obvious that surface divergences cannot be removed by renormalization of some constants in the classical action because the latter does not have the surface terms. Such

terms could appear in result of a specific matter distribution over the horizon Σ , which is not the case of classical black holes. Consequently, to remove such a kind of singularities one should introduce in the effective action a surface functional vanishing at $\beta = 2\pi$. In scalar theory its most general structure should be

$$W_{surf}(\beta) = b_0(2\pi - \beta) \int_{\Sigma} + \beta \int_{\Sigma} (b_1 c_1(\beta) + b_2 c_1(\beta) R_{\Sigma} + (b_3 c_1(\beta) + b_4 c_2(\beta)) R_{\mu\nu} n_i^{\mu} n_i^{\nu} + (b_5 c_1(\beta) + b_6 c_2(\beta)) R_{\mu\nu\lambda\rho} n_i^{\mu} n_i^{\lambda} n_j^{\nu} n_j^{\rho}) \quad (3.14)$$

where b_i are undetermined constants that are not depend on the background geometry.

A remark about the coefficient b_0 is in order. The reason for this term is in renormalization of the gravitational constant G . Indeed, the classical Einstein action should be determined by the integral curvature (2.2) and it includes an additional contribution from the conical singularity. On the other hand, due to the heat coefficient a_1 the divergent term in (2.12) depends only on the local scalar curvature R . Thus this divergence is removed by the renormalization both G and b_0 as follows

$$\begin{aligned} -\frac{1}{16\pi G} \int_{M_{\beta}} \bar{R} + b_0(2\pi - \beta) \int_{\Sigma} + \frac{m^2(1/6 - \xi)}{32\pi^2\epsilon} \int_{M_{\beta}} R = \\ = -\frac{1}{16\pi G_{ren}} \int_{M_{\beta}} \bar{R} + b_{0,ren}(2\pi - \beta) \int_{\Sigma} \quad . \end{aligned} \quad (3.15)$$

Besides, in the first order of the Planck constant the bare parameters G , b_0 are expressed through the renormalized ones G_{ren} , $b_{0,ren}$ as

$$G = G_{ren} - \frac{1}{\epsilon} \left(\frac{1}{6} - \xi \right) \frac{m^2}{2\pi} G_{ren}^2 \quad , \quad (3.16)$$

$$b_0 = b_{0,ren} + \frac{m^2(1/6 - \xi)}{16\pi^2\epsilon} \quad (3.17)$$

and renormalization of G is not influenced by the surface divergences.

To get rid off the surface divergences, $W_{div, surf}$, eq. (2.13), other constants b_i in W_{surf} are renormalized similar to b_0 in such a way that renormalization equations do not depend on the background metric. The number of these constants corresponds to a freedom in the renormalization recipe. Indeed, the surface action must retain isometry invariance and reparametrization invariance of coordinates on Σ . Hence, in a 4-dimensional theory it can be characterized, apart from a trivial constant, by three invariants R_{Σ} , $R_{\mu\nu} n_i^{\mu} n_i^{\nu}$

and $R_{\mu\nu\lambda\rho}n_i^\mu n_i^\lambda n_j^\nu n_j^\rho$. On the other hand, one should remember that origin of $W_{div,surf}$ is in specific geometry of M_β near Σ , which enables to find out a form of its dependence on β . After the scale transformation of the Euclidean time $\tau = \beta\tau'$ the parameter β appears only in the time component of the metric tensor $g_{\tau'\tau'} = g_{\tau\tau}\beta^{-2}$ and, consequently, the surface terms should have the structure $\beta P(\beta^{-2})$ where $P(\beta^{-2})$ is a polynomial vanishing at $\beta = 2\pi$ (additional factor β comes from the volume element in the action integral). As follows from (2.13), $P(\beta^{-2})$ is a second order polynomial that can be decomposed in $c_1(\beta)$ and $c_2(\beta)$. These arguments and the expressions of $a_{\beta,1}$, $a_{\beta,2}$ (see (2.6), (2.7)) define the functional $W_{surf}(\beta)$ in the form (3.14).

After renormalization one obtains a finite but undetermined contribution to the entropy $S_{surf}(\beta)$ due to $W_{surf}(\beta)$

$$\begin{aligned} S_{surf}(\beta) &= (\beta \frac{\partial}{\partial \beta} - 1) W_{surf}(\beta) = \\ &= -2\pi b_0 \int_{\Sigma} + \beta^{-1} \int_{\Sigma} [\gamma_1 + \gamma_2 R_{\Sigma} + (\gamma_3 + \gamma_4 \beta^{-2}) R_{\mu\nu} n_i^\mu n_i^\nu + (\gamma_5 + \gamma_6 \beta^{-2}) R_{\mu\nu\lambda\rho} n_i^\mu n_i^\lambda n_j^\nu n_j^\rho] \end{aligned} \quad (3.18)$$

where γ_i are some numerical combinations from renormalized constants b_i . Thus at the Hawking temperature $S_{surf}(\beta = 2\pi)$ is characterized by four geometrical terms and four corresponding constants. One term, determined by the horizon area $A = \int_{\Sigma}$, was found and discussed in the literature for some particular cases [1]-[5]. The functional (3.18) indicates also other corrections not reducible to the area and depending on the curvature near the horizon. For instance, for Schwarzschild black hole there is a topological term $\int_{\Sigma} R_{\Sigma}$ in the entropy (cf. [6]), and for the charged holes there are other terms that can be expressed through the mass M and charge Q with the help of (2.10), (2.11). Note also, that in the vacuum state ($\beta \rightarrow \infty$) only the first term in (3.18) contributes to the entropy.

Finally, one can proceed in this way and use (3.18) to calculate the surface corrections to the heat capacity of the system, $C_{surf}(\beta) = -\beta \frac{\partial}{\partial \beta} S_{surf}(\beta)$, and to other thermodynamical quantities.

4 Conclusions and remarks

We presented a general form of the surface divergences in the effective action and entropy on static spaces with the horizons. Our conclusion is that this kind of divergences is not reduced only to the area of the horizon but has a more complicated structure depending on the curvature of the space-time. We point out that renormalization of the gravitational constant and other couplings in the one-loop gravitational action is not sufficient to remove these divergences from the all thermodynamical quantities. To this aim one has to introduce a surface functional (3.14) with a number of constants that might be fixed in a more fundamental theory.

The fact that the entropy corrections (3.18) have a more rich form then the Bekenstein-Hawking entropy is not surprising. It should be borne in mind that the one-loop action in quantum gravity includes the second order terms in the curvature, like those in (2.5), which must be taken into account in evaluation of the entropy. From this point of view it would be interesting to analyze the relation between the coupling constants b_0 , γ_i in (3.18) and those in the one-loop gravitational action. A relevant approach can be found in [6].

A remark is in order about different techniques to be used to renormalize the surface divergences. We employed here the heat-kernel method. Other approach is to assume that the wave functions vanish within some fixed distance from the horizon. In this case all the integrals stop short before the horizon and turn out to be finite. Such method was initially proposed in [1] as the "brick-wall" model and it is used with some modifications in other papers.

The heat-kernel and the brick-wall methods result to the similar structure of the surface divergences, but they are not quite equivalent. An analysis shows that they give different dependence on temperature of the of the surface terms in free energy. The reason for this is in their topological properties [17]. The heat-kernel approach does not change the topology of the cone. On the other hand, the brick-wall method is equivalent to cutting of the cone tip, which changes the topology to $S^1 \times R^+$. Thus, from this point of view the heat-kernel technique presented here seems to be more preferable, although

for computation of the local renormalized quantities outside the horizon one can use both them.

In the third method that should be mentioned the cone tip is changed to a smooth manifold [6]. It holds the topology and does not contradict to the heat-kernel results. However, the surface divergences must be extracted from the non-local part of the effective action, which complicates the computations.

Note finally, that our consideration was confined to the static spaces. An interesting problem is to analyze the same aspects for the stationary black-hole geometries.

I would like to thank Sergey Solodukhin for valuable remarks and Franco Buccella and Gennaro Miele for warm hospitality during my visit in the University of Naples. This work was supported in part by the grant RFL000 of the International Science Foundation.

References

- [1] G.'t Hooft, Nucl. Phys. **B256** (1985) 727.
- [2] V.Frolov, I. Novikov, Phys. Rev. **D48** (1993) 4545.
- [3] C.Callan and F.Wilczek, *On geometric entropy*, preprint IASSNs-HEP-93/87; J.S.Dowker, Class. Quantum Grav. **11** (1994) L55.
- [4] L.Susskind and J.Uglum, *Black hole entropy in canonical quantum gravity and superstring theory* preprint SU-ITP-94-1.
- [5] A.O.Barvinsky, V.P.Frolov and A.I.Zelnikov, *Wavefunction of a black hole and the dynamical origin of entropy*, preprint Alberta Thy 13-94, gr-qc/9404036.
- [6] S.N.Solodukhin, *The conical singularity and quantum corrections to entropy of black hole*, preprint JINR E2-94-246.
- [7] G.Cognola, K.Kirsten, L.Vanzo, Phys. Rev. **D49** (1994) 1029.
- [8] D.V.Fursaev, Class. Quantum Grav. **11** (1994) 1431.

- [9] N.D.Birrell and P.C.W.Davies, *Quantum Fields in Curved Space* (Cambridge Univ. Press, New York 1982).
- [10] D.V.Fursaev and G.Miele, Phys. Rev. **D49** (1994) 987.
- [11] D.V.Fursaev, *Spectral geometry and one-loop divergences on manifolds with conical singularities*, hep-th/9405143, Phys. Lett. B (1994) to be published.
- [12] J.S.Dowker, *Heat kernels on curved cones*, hep-th/9606002, Class. Quant. Grav. (1994) to be published; J.S.Dowker, *Effective actions with fixed points*, preprint MUTP/94/14, hep-th/9406144.
- [13] D.D.Sokolov and A.A.Starobinsky, DAN SSSR **22** (1977) 312, G.Hayward and J.Louko, Phys. Rev. **D42** (1990) 4032.
- [14] M.Bañados, C.Teitelboim, J.Zanelli, Phys. Rev. Lett. **72** (1994) 957.
- [15] J.A.Schouten, *Ricci-Calculus* (Springer-Verlag, Berlin 1954).
- [16] S.W.Hawking, in *General Relativity: an Einstein Century Survey*, edited by S.W.Hawking and W.Israel (Cambridge University Press, Cambridge, England,1979).
- [17] R.Empanan, *Heat kernels and thermodynamics in Rindler space*, preprint EHU-FT-94/5, hep-th/9407064.